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A Two-Dimensional Mean Problem

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Let $0 < r_n < 1$ and $w_n = e^{i2\pi/n}$, $n = 1, 2,\ldots$. For a function f holomorphic in the open unit disc U, we consider the linear functionals s_n defined by the means $s_n(r_n, f) = (1/n) \sum_{k=1}^{n} f(r_n w_n^k)$. If $0 < r_n \leqslant \rho < 1$, we prove that f is uniquely determined by $s_n(r_n, f)$, $n = 1, 2,\ldots$, and in fact, f can be represented by a polynomial series whose coefficients involve $s_n(r_n, f)$. The case $0 < r_n \leqslant 1$ is also considered. In particular, if $r_n = 1$ for all large n, there exist nontrivial functions f, holomorphic in U and continuous on the closure of U, such that $s_n(r_n, f) = 0$ for $n = 1, 2,\ldots$.

1. INTRODUCTION AND MAIN RESULTS

Let U denote the open unit disc in the complex plane with closure $\overline{U}$ and boundary T. Let $H = H(U)$ denote the space of functions holomorphic in U; and as usual, let H^p be the Hardy spaces and A the space of functions in H which are continuous on $\overline{U}$. For each positive integer n, let $w_n^k = \exp(i2\pi k/n)$, $k = 1,\ldots, n$, be the nth roots of unity. For a continuous function f on T, we consider its arithmetic means

$$s_n(f) = \frac{1}{n} \sum_{k=1}^{n} f(w_n^k).$$

These are Riemann sums and hence converge to the Riemann integral

$$s_\infty(f) = \int_0^1 f(e^{i2\pi t})\, dt$$

of f as $n \to \infty$. The sequence $r_n(f) = s_n(f) - s_\infty(f)$, called the sequence of Riemann coefficients of f in [4], has similar asymptotic behavior to the sequence of Fourier coefficients of f for certain classes of functions f (cf. [4, 7]). Since the Fourier coefficients of f uniquely determine f, it is natural to ask if

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the Riemann coefficients of f would also uniquely determine f. However, it is clear that any "odd" function

$$f(z) = \sum_{k=1}^{\infty} a_k(z^k - z^{-k}),$$

where $\sum | a_k | < \infty$ say, satisfies

$$s_n(f) = 0, \qquad n = 1, 2, \ldots . \tag{1}$$

Hence, we only consider functions holomorphic in U. This problem was studied in [2], [6], and [8]. We collect some of the known results in the following

THEOREM A. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$ *be in* A *such that* (1) *is satisfied. Then* f *is the zero function, if one of the following conditions is satisfied*:

(a) $f' \in H^1$;

(b) $a_n = O(1/n^{1+\epsilon})$ *for some* $\epsilon > 0$;

(c) $\sum_{k=N}^{\infty} | a_k | = O(1/N)$; *or*

(d) $f(z) = \sum_{n=0}^{\infty} b_n z^{q^n}$ *with* $\sum | b_n | < \infty$ *where* q *is some positive integer.*

Of course each of the above sufficient conditions is a technical one. However, it will be shown in Section 4 that there exists a nontrivial $f(z) = \sum a_n z^n$ in A with $| a_n | \leqslant 1/n$ for all n such that (1) is satisfied.

We remark that the problem considered above is a "one-dimensional" one. In [8], a "two-dimensional" problem was posed, and it is the intention of this paper to study it. Let $0 < r_n < 1$, $n = 1, 2, \ldots$. For each $f \in H$, consider the "two-dimensional" means

$$s_n(r_n, f) = \frac{1}{n} \sum_{k=1}^{n} f(r_n w_n{}^k)$$

of f, that is, the means taken on the concentric circles $| z | = r_n$, $n = 1, 2, \ldots$. We will establish the following.

THEOREM 1. *Let* $0 < r_n \leqslant \rho < 1$, $n = 1, 2, \ldots$, *and let* $f \in H$ *satisfy*

$$s_n(r_n, f) = 0, \qquad n = 1, 2, \ldots . \tag{2}$$

Then f *is the zero function.*

It will be clear (from the following Theorem 2) that none of the r_n's in Theorem 1 can be replaced by 0. The condition that the r_n's are uniformly bounded away from 1 is a technical one. We will give a proposition in Section 4 where the r_n's are allowed to tend to 1. If some r_n's would be 1, then to define the means $s_n(r_n, f)$, we would have to assume that f is a function in A. However, we will show that there is a nontrivial function f in A with $s_n(r_n, f) = 0$ for all $n = 1, 2, \ldots$, where all, with the exception of a finite number of the r_n's, are equal to 1. The next results show that if any of the conditions in (2) is omitted, then Theorem 1 no longer holds.

THEOREM 2. *Let $0 < r_n \leqslant 1$, $n = 1, 2, \ldots$. For each positive integer N, there is a unique polynomial P_N of degree N, leading coefficient equal to 1, and $P_N(0) = 0$, such that*

$$s_n(r_n, P_N) = r_n{}^n \delta_{n,N}, \qquad n = 1, 2, \ldots \tag{3}$$

where, as usual, $\delta_{n,N}$ is the Kronecker delta.

The polynomials P_N can be found explicitly and will be studied in Section 3. When $0 < r_n \leqslant \rho < 1$, Theorem 1 tells us that each function f holomorphic in U is uniquely determined by its means $s_n(r_n, f)$. This leads to the following interesting, and perhaps important, question: How do we reconstruct a function $f \in H$ from its means $s_n(r_n, f)$? From Theorem 4 below, we will see that f can be reconstructed from a polynomial series whose coefficients are the means $s_n(r_n, f)$. The "one-dimensional" problem has been studied in [5], and the representation polynomial series there is called a "Riemann series." In Section 3, we will prove the following results.

THEOREM 3. *Let $0 < r_n \leqslant \rho < 1$, $n = 1, 2, \ldots$ and $\{\alpha_n\}$ be a sequence of complex numbers which satisfies the condition*

$$\limsup_{n\to\infty} |\alpha_n|^{1/n}/r_n \leqslant 1. \tag{4}$$

Then the polynomial series

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{r_n{}^n} P_n(z), \tag{5}$$

where the polynomials P_n are defined in Theorem 2, converges uniformly on every compact subset of U to a function $f \in H$, such that $s_n(r_n, f) = \alpha_n$ for each $n = 1, 2, \ldots$.

THEOREM 4. *Let $0 < r_n \leqslant \rho < 1$ and P_n be the polynomials defined in Theorem 2. Then every function f holomorphic in U can be represented by a polynomial series:*

$$f(z) = f(0) + \sum_{n=1}^{\infty} \frac{s_n(r_n, f) - f(0)}{r_n{}^n} P_n(z) \tag{6}$$

where the series converges uniformly on every compact subset of U to f.

We call (5) a "two-dimensional" Riemann series, and (6) a "two-dimensional" Riemann series expansion of f.

In Section 4, we will study the case when the radii r_n are allowed to approach 1.

2. Proof of Theorem 1

We will first prove $f(0) = 0$.

LEMMA 1. *Let* $0 < r_n \leqslant \rho < 1$, $n = 1, 2,\ldots, f \in H$, *and* $s_n(r_n, f) = 0$ *for infinitely many* n. *Then* $f(0) = 0$.

Proof. By choosing a subsequence, if necessary, we may assume that $s_n(r_n, f) = 0$ for $n = n_j$, $j = 1, 2,\ldots,$ and $r_{n_j} \to r_0 \leqslant \rho < 1$. Then for $n = n_j$, we have

$$\begin{aligned} |f(0)| &= |f(0) - s_n(r_n, f)| \\ &= \left| \int_0^1 f(r_0 e^{i2\pi t})\, dt - s_n(r_n, f) \right| \\ &\leqslant \left| \int_0^1 f(r_0 e^{i2\pi t})\, dt - \frac{1}{n} \sum_{k=1}^{n} f(r_0 e^{i2\pi k/n}) \right| \\ &\quad + \frac{1}{n} \sum_{k=1}^{n} | f(r_n e^{i2\pi k/n}) - f(r_0 e^{i2\pi k/n})|. \end{aligned}$$

The first term on the right tends to zero because Riemann sums converge to the Riemann integral and the second term is arbitrarily small for large $n = n_j$ because f is uniformly continuous on $|z| \leqslant (1 + r_0)/2$. Hence, $f(0) = 0$.

In virtue of Lemma 1, we may now write

$$f(z) = \sum_{k=1}^{\infty} a_k z^k,$$

so that

$$s_n(r_n, f) = \sum_{k=1}^{\infty} a_k r_n{}^k \left\{ \frac{1}{n} \sum_{j=1}^{n} w_n^{jk} \right\} = \sum_{k=1}^{\infty} a_{kn} r_n^{kn}.$$

From hypothesis (2), it is necessary and sufficient to prove that the infinite homogeneous system

$$\sum_{k=1}^{\infty} r_n^{(k-1)n} a_{kn} = 0, \qquad n = 1, 2,\ldots \tag{7}$$

has only the trivial solution. Let $F = (F_{i,j})$ be the (infinite) coefficient matrix:

$$F = (F_{i,j}) = \begin{bmatrix} 1 & r_1 & r_1^2 & r_1^3 & r_1^4 & r_1^5 & r_1^6 & \cdots \\ 0 & 1 & 0 & r_2^2 & 0 & r_2^4 & 0 & \cdots \\ 0 & 0 & 1 & 0 & 0 & r_3^3 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \tag{8}$$

where

$$\begin{aligned} F_{i,j} &= 0 \qquad \text{if} \quad i \nmid j \\ &= r_i^{j-i} \qquad \text{if} \quad i \mid j, \end{aligned} \tag{9}$$

and let $F_N = (F_{i,j})_{1 \leqslant i,j \leqslant N}$, $N = 1, 2,...$, be the truncated $N \times N$ matrices. For each N, we are interested to find the inverse G_N of F_N. From the properties of F, it is easy to show that the matrices $G_N = (g_i(j))_{1 \leqslant i,j \leqslant N}$, $N = 1, 2,...$, are truncations of an infinite matrix

$$G = (g_i(j)) = \begin{bmatrix} g_1(1) & g_1(2) & \cdots \\ g_2(1) & g_2(2) & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Indeed, $G_N F_N = I_N$ means $\sum_{l=1}^{N} g_k(l) F_{l,n} = \delta_{k,n}$, $1 \leqslant k$, $n \leqslant N$, and by using (9), we have

$$\sum_{l \mid n} g_k(l)\, r^{n-l} = \delta_{k,n}\,. \tag{10}$$

In particular, the $g_k(l)$'s have the following properties:

$$g_1(1) = 1, \qquad g_1(n) = -\sum_{\substack{d \mid n \\ d < n}} g_1(d)\, r_d^{n-d}, \tag{11}$$

and

$$g_k(n) = 0 \qquad \text{if} \quad k \nmid n. \tag{12}$$

Here, (11) follows trivially from (10), and (12) can be obtained by an induction proof as follows. Indeed, if $k \nmid n$, then from (10) it follows that

$$g_k(n) + \sum_{\substack{d \mid n \\ d < n}} g_k(d)\, r_d^{n-d} = \delta_{k\,n} = 0.$$

If $d \mid n$ and $d < n$ then $k \nmid d$ (for otherwise $k \mid n$) so that $g_k(d) = 0$ by the induction hypothesis. Hence, (12) is obtained. From (10) and (12), we also have

$$g_k(1) = 0 \quad \text{if} \quad k > 1, \qquad \text{and} \qquad g_k(k) = 1 \quad \text{for} \quad k \geqslant 1. \tag{13}$$

Next, for a fixed integer k, $k \geqslant 1$, we define

$$h_k(n) = g_k(nk) \qquad \text{and} \qquad \rho_l = r_{kl}^k . \tag{14}$$

Then from (12) and (13), it is clear that $h_k(n)$ satisfies

$$\begin{aligned} & h_k(1) = 1 \\ \text{and} \qquad & h_k(n) = -\sum_{\substack{l \mid n \\ l<n}} h_k(l)\, \rho_l^{n-l} \qquad \text{if} \quad n > 1. \end{aligned} \tag{15}$$

We remark that the h_k's satisfy the same recursive scheme as g_1 with r_n replaced by ρ_n. To estimate g_1 and h_k, we need the following combinatorial lemma.

LEMMA 2. *Let H be a function defined on the set of positive integers by*

$$\begin{aligned} & H(1) = 1 \\ \textit{and} \qquad & H(n) = \sum_{\substack{l \mid n \\ l<n}} H(l) \qquad \textit{if} \quad n > 1. \end{aligned}$$

Then $H(n) \leqslant 2^{(\log n/\log 2)^2}$ for all n.

Proof. As usual, let $\Omega(n)$ denote the number of prime factors of n, counted with their multiplicities (cf. [10]). We will call $\Omega(n)$ the length of n. Now, if p is a prime number, then by definition $H(p) = H(1) = 1$, $H(p^2) = H(p) + H(1) = 2, \ldots, H(p^j) = H(p^{j-1}) + \cdots + H(1) = 2^{j-1}, \ldots$. Hence, in general, if $p_1, \ldots, p_t$ are primes and $\alpha_1, \ldots, \alpha_t$ are positive integers, then $H(p_1^{\alpha_1} \cdots p_t^{\alpha_t})$ does not depend on $p_1, \ldots, p_t$ but only depends on $\alpha_1, \ldots, \alpha_t$. Also for each positive integer k, there are only a finite number of ways to choose positive integers $\alpha_1, \ldots, \alpha_t$ such that $\alpha_1 + \cdots + \alpha_t = k$. We can therefore define

$$S_k = \max\{H(n) \colon \Omega(n) = k\}.$$

(Here, we note that if $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, then $\Omega(n) = \alpha_1 + \cdots + \alpha_t = k$.) It is clear that $S_1 \leqslant S_2 \leqslant \cdots$. Let us write $n = p_1 \cdots p_k$ where some of the primes p_i's may be equal, so that $\Omega(n) = k$. The number of factors of n with length $k - 1$ is at most $k = \binom{k}{1}$, the number of factors of n with length $k - 2$ is at most $\binom{k}{2}, \ldots$. Hence, we have

$$\begin{aligned} S_k &\leqslant \binom{k}{1} S_{k-1} + \binom{k}{2} S_{k-2} + \cdots + \binom{k}{k} S_0 \\ &\leqslant S_{k-1} \left[\binom{k}{1} + \cdots + \binom{k}{k} \right] \leqslant 2^k S_{k-1}, \end{aligned}$$

and therefore,

$$S_k \leqslant 2^k S_{k-1} \leqslant 2^k 2^{k-1} S_{k-2} \leqslant \cdots \leqslant 2^{k+(k-1)+\cdots+1} \leqslant 2^{k^2}.$$

Thus, if n is any positive integer, with length $\Omega(n) = k$ say, then

$$H(n) \leqslant 2^{(\Omega(n))^2}.$$

But $n = p_1 \cdots p_k \geqslant 2^k = 2^{\Omega(n)}$. This completes the proof of the lemma.

LEMMA 3. *Let $0 < r_n \leqslant 1$. Then for each n, $|g_1(n)| \leqslant H(n)$.*

Proof. We have $g_1(1) = H(1) = 1$. Hence, from (11) and by using the induction hypothesis, we have

$$|g_1(n)| \leqslant \sum_{\substack{d \mid n \\ d<n}} |g_1(d)|\, r_d^{n-d} \leqslant \sum_{\substack{d \mid n \\ d<n}} H(d) = H(n)$$

for $n > 1$.

If the r_n's are uniformly bounded above by $\rho \leqslant 1$, then we have the following upper bound for $g_1(n)$.

LEMMA 4. *Let $0 < r_n \leqslant \rho \leqslant 1$. Then for all $n = 2, 3, \ldots$,*

$$|g_1(n)| \leqslant \rho^{n/2}\, 2^{(\log n/\log 2)^2}.$$

Proof. We have, from (11) and by using Lemma 3, for $n > 1$,

$$|g_1(n)| \leqslant \sum_{\substack{d \mid n \\ d<n}} |g_1(d)|\, r_d^{n-d} \leqslant \sum_{\substack{d \mid n \\ d<n}} H(d)\, \rho^{n-d}.$$

Since $d < n$ and $d \mid n$ imply $d \leqslant n/2$, we have $n - d \geqslant n/2$, so that $\rho^{n-d} \leqslant \rho^{n/2}$. By Lemma 2, we then obtain

$$|g_1(n)| \leqslant \rho^{n/2} \sum_{\substack{d \mid n \\ d<n}} H(d) = \rho^{n/2} H(n) \leqslant \rho^{n/2}\, 2^{(\log n/\log 2)^2}.$$

By using the above argument and (15), we also have

LEMMA 5. *Let $0 < r_n \leqslant \rho \leqslant 1$ and k be any positive integer. Then*

$$|g_k(nk)| = |h_k(n)| \leqslant \rho^{nk/2}\, 2^{(\log n/\log 2)^2}, \qquad n > 1. \tag{16}$$

We are now ready to complete the proof of Theorem 1. Let $0 < r_n \leqslant \rho < 1$, and let k be any positive integer. From (7) and (8), we have

$$0 = [g_k(1),\ldots, g_k(N) \mid 0,\ldots] \left[\begin{array}{c|c} F_N & R \\ \hline 0 & S \end{array}\right] \left[\begin{array}{c} a_1 \\ \vdots \\ a_N \\ \hline a_{N+1} \\ \vdots \end{array}\right]$$

and hence,

$$0 = [[g_k(1),\ldots, g_k(N)]\, F_N \mid [g_k(1),\ldots, g_k(N)]\, R] \left[\begin{array}{c} a_1 \\ \vdots \\ a_N \\ \hline a_{N+1} \\ \vdots \end{array}\right]$$

$$= [0,\ldots, 0, 1, 0,\ldots, 0 \mid [g_k(1),\ldots, g_k(N)]\, R] \left[\begin{array}{c} a_1 \\ \vdots \\ a_N \\ \hline a_{N+1} \\ \vdots \end{array}\right],$$

where the 1 occurs at the kth entry and we have used the fact that $G_N F_N = I_N$. Hence, we have

$$a_k = -\sum_{j=N+1}^{\infty} a_j \left(\sum_{\substack{d \mid j \\ d \leqslant N}} g_k(d)\, F_{d,j} \right) = -\sum_{j=N+1}^{\infty} a_j c_j, \tag{17}$$

where, using (9),

$$c_j = \sum_{\substack{d \mid j \\ d \leqslant N}} g_k(d)\, F_{d,j} = \sum_{\substack{d \mid j \\ d \leqslant N}} g_k(d)\, r_d^{j-d}. \tag{18}$$

From (12), we see that $c_j = 0$ if $k \nmid j$. Thus, (17) can be written as

$$a_k = -\sum_{l \geqslant (N+1)/k}^{\infty} c_{lk} a_{lk}. \tag{19}$$

Now, in (18), applying (12), (11), (14), (15) and Lemma 5, we have

$$\begin{aligned}
| c_{lk} | &= \Big| \sum_{\substack{d \mid lk \\ d \geqslant N}} g_k(d)\, r_d^{lk-d} \Big| \\
&= \Big| \sum_{\substack{\nu \mid l \\ \nu \leqslant N/k}} g_k(k\nu)\, r_{k\nu}^{k(l-\nu)} \Big| \\
&= \Big| g_k(k)\, r_k^{(l-1)k} + \sum_{\substack{\nu \mid l \\ 1<\nu \leqslant N/k}} h_k(\nu)\, \rho_\nu^{l-\nu} \Big| \\
&\leqslant \rho^{(l-1)k} + \sum_{\substack{\nu \mid l \\ 1<\nu \leqslant N/k}} \rho^{k\nu/2}\, 2^{(\log\nu/\log 2)^2} \rho^{k(l-\nu)} \\
&= \rho^{(l-1)k} + \sum_{\substack{\nu \mid l \\ 1<\nu \leqslant N/k}} \rho^{k(l-\nu/2)}\, 2^{(\log\nu/\log 2)^2}.
\end{aligned}$$

But $\nu \mid l$, $\nu < l$ implies that $\nu \leqslant l/2$. Hence, for $l \geqslant 2$,

$$| c_{lk} | \leqslant \rho^{kl/2} \Big(1 + \sum_{\substack{\nu \mid l \\ 1<\nu \leqslant N/k}} 2^{(\log\nu/\log 2)^2} \Big).$$

Therefore, for $lk \geqslant N + 1$, and sufficiently large N, we have

$$| c_{lk} | \leqslant \rho^{lk/2} \Big(1 + \frac{N}{k}\, 2^{(\log N)^2} \Big) < \rho^{lk/4}. \tag{20}$$

Now since the power series $\sum a_k z^k$ has radius of convergence $\geqslant 1$, we have

$$\sum_{j=1}^{\infty} | a_j |\, \rho^{j/4} < \infty,$$

so that combining (19) and (20) and taking $N \to \infty$, we can conclude that $a_k = 0$. This holds for every k. That is, the given function $f \in H$, satisfying (2), is the zero function.

3. Two-Dimensional Riemann Series Representation

In this section we will prove Theorems 2, 3, and 4. Let $P_N(z) = a_1 z + \cdots + a_N z^N$ be a polynomial of degree N. That P_N satisfies (3) means that

$$\sum_{1 \leqslant k \leqslant N/n} a_{kn} r_n^{(k-1)n} = \delta_{N,n}, \qquad n = 1, 2, \ldots .$$

That is, the coefficients $a_1, \ldots, a_N$ of P_N are uniquely determined by the nonhomogeneous system

$$F_N \begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} a_1 \\ \vdots \\ a_N \end{bmatrix} = G_N \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \tag{21}$$

Since $g_N(N) = 1$, we have $a_N = 1$. This completes the proof of Theorem 2.

From (21) we note that

$$\begin{aligned} a_n = g_n(N) &= 0 && \text{if} \quad n \nmid N \\ &= h_n(N/n) && \text{if} \quad n \mid N. \end{aligned}$$

Hence, we can write

$$P_N(z) = \sum_{n \mid N} h_n\left(\frac{N}{n}\right) z^n = \sum_{n \mid N} g_n(N)\, z^n. \tag{22}$$

For reference we list the first six polynomials:

$$\begin{aligned} P_1(z) &= z, \\ P_2(z) &= -r_1 z + z^2, \\ P_3(z) &= -r_1^2 z + z^3, \\ P_4(z) &= (-r_1^3 + r_1 r_2^2)\, z - r_2^2 z^2 + z^4, \\ P_5(z) &= -r_1^4 z + z^5, \\ P_6(z) &= (-r_1^5 + r_1 r_2^4 + r_1^2 r_3^3)\, z - r_2^4 z^2 - r_3^3 z^3 + z^6. \end{aligned} \tag{23}$$

Suppose now $0 < r_n \leqslant \rho < 1$ for all n. From Lemma 5, we have

$$\begin{aligned} | P_n(z)| &\leqslant | z |^n + \sum_{\substack{k \mid n \\ k < n}} | g_k(n)|\, | z |^k \\ &\leqslant | z |^n + \sum_{\substack{k \mid n \\ k < n}} \rho^{n/2}\, 2^{((\log n/k)/\log 2)^2} | z |^k \\ &\leqslant | z |^n + \rho^{n/2}\, 2^{(\log n/\log 2)^2} | z |^{n/2}\, d(n), \end{aligned}$$

where, as usual, $d(n)$ denotes the number of divisors of n (cf. [10]).

Let $\rho^{1/2} \leqslant r < 1$. Then for all z with $|z| \leqslant r$, we have

$$|P_n(z)| \leqslant 2r^{n/2}$$

for all large n. Hence, if $\{\alpha_n\}$ is any sequence satisfying (4), then

$$\limsup_{n\to\infty} \left| \frac{\alpha_n}{r_n{}^n} P_n(z) \right|^{1/n} \leqslant r^{1/2} < 1$$

uniformly for $|z| \leqslant r$. This proves that the polynomial series (5) converges uniformly on every compact subset of U to some function $f \in H$. Write

$$f(z) = \sum_{n=1}^{\infty} \frac{\alpha_n}{r_n{}^n} P_n(z).$$

Then by Theorem 2, we have

$$S_N(r_N, f) = \sum_{n=1}^{\infty} \frac{\alpha_n}{r_n{}^n} S_N(r_N, P_n)$$

$$= \sum_{n=1}^{\infty} \frac{\alpha_n}{r_n{}^n} r_N{}^N \delta_{n,N} = \alpha_N,$$

$N = 1, 2, \ldots$. This completes the proof of Theorem 3.

We now proceed to prove Theorem 4. Let $f = \sum a_n z^n \in H$ and let $\alpha_n = s_n(r_n, f) - f(0)$. Then

$$\frac{\alpha_n}{r_n{}^n} = a_n + \sum_{\nu=2}^{\infty} a_{\nu n} r_n^{(\nu-1)n}.$$

Hence,

$$\frac{|\alpha_n|}{r_n{}^n} \leqslant |a_n| + \sum_{\nu=2}^{\infty} |a_{\nu n}| \rho^{(\nu-1)n}$$

$$\leqslant |a_n| + \sum_{\nu=2}^{\infty} |a_{\nu n}| \tilde{\rho}^{\nu n}$$

$$\leqslant |a_n| + \sum_{\nu=0}^{\infty} |a_\nu| \tilde{\rho}^{\nu},$$

where $\tilde{\rho} = \rho^{1/2} < 1$. The infinite series on the right converges since $f \in H$. Also, since $f \in H$, $\limsup |a_n|^{1/n} \leqslant 1$. Hence, $\limsup |\alpha_n|^{1/n}/r_n \leqslant 1$, and by Theorem 3, the polynomial series

$$f(0) + \sum_{n=1}^{\infty} \frac{\alpha_n}{r_n{}^n} P_n(z) = f(0) + \sum_{n=1}^{\infty} \frac{s_n(r_n, f) - f(0)}{r_n{}^n} P_n(z)$$

converges uniformly on every compact subset of U to a function $F \in H$. Clearly, $s_n(r_n, F) = s_n(r_n, f)$, or $s_n(r_n, F - f) = 0$, for $n = 1, 2, \ldots$. By Theorem 1, $F \equiv f$. This completes the proof of Theorem 4.

4. Extensions and Counterexamples

In this section we will consider the case when the radii r_n are allowed to approach 1, and we will show that in general we cannot take r_n to be 1 for all large n. We need a lemma first.

LEMMA 6. *Let* $0 < r_n \leqslant (\frac{1}{2})^{1/n}$ *for* $n = 1, 2, \ldots$. *Then* $|g_1(n)| \leqslant \frac{1}{2}$ *for* $n = 2, 3, \ldots$.

Proof. We know from (11) that $g_1(2) = -r_1 g_1(1) = -r_1$ so that $|g_1(2)| \leqslant \frac{1}{2}$. We will prove the general result by induction. Again by (11) we have

$$|g_1(n)| \leqslant \sum_{\substack{d|n \\ d<n}} |g_1(d)|\, r_d^{n-d} \leqslant \sum_{\substack{d|n \\ d<n}} |g_1(d)|\, (\tfrac{1}{2})^{(n-d)/d}$$

$$\leqslant (\tfrac{1}{2})^{n-1} + \sum_{\substack{d|n \\ 1<d<n}} |g_1(d)|\, (\tfrac{1}{2})^{(n-d)/d}$$

$$\leqslant (\tfrac{1}{2})^{n-1} + \sum_{\substack{d|n \\ 1<d<n}} (\tfrac{1}{2})^{n/d},$$

where the last inequality follows from the induction hypothesis. Since $n > 2$, we have $(n - 1) \neq n/d$ for all $d \mid n$ and $1 < d < n$. Hence,

$$|g_1(n)| \leqslant (\tfrac{1}{2})^{n-1} + \sum_{\substack{d|n \\ 1<d<n}} (\tfrac{1}{2})^{n/d}$$

$$< \sum_{d=2}^{\infty} (\tfrac{1}{2})^d = \tfrac{1}{2}.$$

With the above lemma and the results developed in Section 2, we can now prove the following.

PROPOSITION 1. *Let* $0 < r_n \leqslant 1$ *such that for all large n, say* $n > n_0$,

$$r_n \leqslant (\tfrac{1}{2})^{1/n}. \tag{24}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *such that* $\sum |a_n| < \infty$. *Then f must be the zero function if* $s_n(r_n, f) = 0$ *for each* $n = 1, 2, \ldots$.

Proof. Since $\sum | a_n | < \infty$, we know that $f \in A$ so that $s_n(1, f)$ is defined. By a proof similar to that of Lemma 1, we can also conclude that $f(0) = 0$. Let k be any positive integer. Fix k. Then for any arbitrarily large positive integer N, we have, from (19) in Section 2,

$$| a_k | \leqslant \sum_{l \geqslant (N+1)/k} | c_{lk} | \, | a_{lk} |, \tag{25}$$

where, as in Section 2,

$$| c_{lk} | = \Big| \rho_1^{(l-1)} + \sum_{\substack{\nu | l \\ 1 < \nu \leqslant N/k}} h_k(\nu) \, \rho_\nu^{l-\nu} \Big|$$

where $h_k(\nu) = g_k(\nu k)$ and $\rho_\nu = r_{\nu k}^k$. Let us first assume that (24) holds for all $n = 1, 2, \ldots$. Then since h_k satisfies the same recursive scheme as g_1 with r_k replaced by ρ_k (cf. (11) and (15)), we see from Lemma 6 that $| h_k(\nu)| \leqslant \frac{1}{2}$ for all ν. Hence,

$$\begin{aligned} | c_{lk} | &\leqslant r_k^{k(l-1)} + \tfrac{1}{2} \sum_{\nu | l} r_{\nu k}^{(l-\nu)k} \\ &\leqslant (\tfrac{1}{2})^{(l-1)} + \tfrac{1}{2} \sum_{\nu | l} (\tfrac{1}{2})^{(l-\nu)/\nu k} \\ &< 1 + \frac{1}{2} \frac{1}{1 - (\frac{1}{2})^{1/k}} \equiv C_k . \end{aligned}$$

Putting this into (25), we have

$$| a_k | \leqslant C_k \sum_{\nu = N+1}^{\infty} | a_\nu | \to 0 \qquad \text{as} \quad N \to \infty.$$

Hence, $a_k = 0$ for each k or $f \equiv 0$. More generally, suppose now (24) is satisfied for $n > n_0$. Let

$$F(z) = f(z) - \sum_{k=1}^{n_0} 2^k s_k(\tfrac{1}{2}, f) \, \tilde{P}_k(z)$$

where the polynomials $\tilde{P}_k$ are defined in Theorem 2 for the sequence $\frac{1}{2}, \ldots, \frac{1}{2}$ (so that $s_k(\frac{1}{2}, \tilde{P}_j) = \delta_{k,j}/2^j$, $1 \leqslant j, k \leqslant n_0$). Hence,

$$\begin{aligned} s_n(\tfrac{1}{2}, F) &= s_n(\tfrac{1}{2}, f) - \sum_{k=1}^{n_0} 2^k s_k(\tfrac{1}{2}, f) \, \delta_{n,k}/2^n \\ &= s_n(\tfrac{1}{2}, f) - s_n(\tfrac{1}{2}, f) = 0 \end{aligned}$$

for $n = 1, \ldots, n_0$. But for $n > n_0$, $s_n(r_n, F) = s_n(r_n, f) = 0$. Hence, from the above conclusion with the sequence $\frac{1}{2}, \ldots, \frac{1}{2}$, $r_{n_0+1}, \ldots$ (which clearly satisfies (24) for all n), we can conclude that $F \equiv 0$, or $f(z) = a_1 z + \cdots + a_{n_0} z^{n_0}$, a polynomial of degree at most n_0. But then $0 = s_{n_0}(r_{n_0}, f) = a_{n_0} r_{n_0}^{n_0}$, $0 = s_{n_0-1}(r_{n_0-1}, f) = a_{n_0-1} r_{n_0-1}^{n_0-1}, \ldots, 0 = s_1(r_1, f) = a_1 r_1$. Hence, f is the zero function as asserted. This completes the proof of Proposition 1. A similar transformation can be used to derive the following from Theorem 1:

COROLLARY 1. *Let* $0 < r_n \leqslant 1$, $n = 1, 2, \ldots$ *and* $\limsup_{n\to\infty} r_n < 1$. *Let* $f \in H$ *satisfy* $s_n(r_n, f) = 0$ *for* $n = 1, 2, \ldots$. *Then* f *is the zero function.*

Next, we have the following result concerning "two-dimensional" Riemann series expansion.

PROPOSITION 2. *Let* $0 < r_n \leqslant 1$ *such that* $r_n \leqslant (\frac{1}{2})^{1/n}$ *for all large* n. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *satisfy*

$$a_n = O(n^{-(1+\epsilon)}) \qquad \text{for some} \quad \epsilon > 0. \tag{26}$$

Then f *can be represented by the Riemann series expansion* (6) (*uniformly on every compact subset of* U).

Proof. Let

$$F(z) = \sum_{n=0}^{\infty} \frac{s_n(r_n, f) - f(0)}{r_n{}^n} P_n(z) + f(0) = \sum_{n=0}^{\infty} b_n z^n$$

where $P_n(z) = \sum_{\nu | n} g_\nu(n) z^\nu$. From the estimate in Lemma 6 (where we can assume without loss of generality by the transformation used at the end of the above proof that $r_n \leqslant (\frac{1}{2})^{1/n}$ for all n), it is clear that the series converges uniformly on every compact subset of U to $F \in H$. Also, it is clear that $s_n(r_n, F) = s_n(r_n, f)$ for $n = 1, 2, \ldots$. In order to apply Proposition 1 to conclude that $F \equiv f$, it is sufficient to prove that $\sum | b_n | < \infty$. For $n \geqslant 1$, it is easy to see that

$$\begin{aligned} b_n &= \sum_{\nu=1}^{\infty} \frac{s_{\nu n}(r_{\nu n}, f) - f(0)}{r_{\nu n}^{\nu n}} g_n(\nu n) \\ &= \sum_{k=1}^{\infty} a_{kn} r_n^{(k-1)n} + \sum_{\nu=2}^{\infty} g_n(\nu n) \sum_{k=1}^{\infty} a_{k\nu n} r_{\nu n}^{(k-1)\nu n}. \end{aligned}$$

Hence, for all large n, we have $| a_j | \leqslant c/j^{1+\epsilon}$ and $| g_n(j) | \leqslant \frac{1}{2}$ (where we again apply Lemma 6, by assuming without loss of generality that (24) holds for all n), so that

$$| b_n | \leqslant \frac{c}{n^{1+\epsilon}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \frac{1}{2^{k-1}} + \frac{c}{2n^{1+\epsilon}} \sum_{\nu=2}^{\infty} \frac{1}{\nu^{1+\epsilon}} \sum_{k=1}^{\infty} \frac{1}{k^{1+\epsilon}} \frac{1}{2^{k-1}}.$$

Hence, $\sum | b_n | < \infty$ and we have completed the proof of Proposition 2.

We will now show that one cannot expect a very general result. In a private communication [1], Ching (who unfortunately passed away in 1974 at the age of 27) has observed that the function

$$f(z) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} z^n, \tag{27}$$

where μ is the classical number theoretic Möbius function (cf. [10]), is in A, and satisfies the condition $s_n(1, f) = 0$ for all $n = 1, 2, \ldots$. It is obvious that the function f in (27) is holomorphic in U. To prove that f is continuous on $\bar{U}$, we can use the following estimate of Davenport [9]

$$\sum_{k=1}^{n} \mu(k) e^{ik\theta} = O(n(\log n)^{-2}), \tag{28}$$

where the estimate is uniform in θ, and apply the standard technique of summation by parts to the partial sums of the series (27). To prove that $s_n(1, f) = 0$ for all n, it is necessary and sufficient to prove that

$$\sum_{k=1}^{\infty} \frac{\mu(kn)}{k} = 0, \qquad n = 1, 2, \ldots. \tag{29}$$

It is well known (cf. [12]) that (29) holds for $n = 1$. For $n > 1$, we have:

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{\mu(kn)}{k} &= \sum_{(k,n)=1} \frac{\mu(kn)}{k} = \mu(n) \sum_{(k,n)=1} \frac{\mu(n)}{n} \\ &= \mu(n) \prod_{p \nmid n} \left(1 - \frac{1}{p}\right) = \mu(k) \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \Big/ \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \\ &= 0.\end{aligned}$$

By using the transformation technique at the end of the proof of Proposition 2 and the example in (27), we can conclude the following:

PROPOSITION 3. *Let* $0 < r_n \leqslant 1$ *and* $r_n = 1$ *for all large n. There exists a nontrivial function* $f \in A$ *such that* $s_n(r_n, f) = 0$ *for* $n = 1, 2, \ldots$.

5. FINAL REMARKS

In this paper, when the radii r_n are uniformly bounded away from 1, the two-dimensional problem is completely solved. If the radii r_n are allowed to tend to 1, both positive and negative results are obtained in Section 4.

However, it is clear that there is still a big gap between these results. The functions $g_n(k)$ introduced in this paper take the place of the number theoretic Möbius function $\mu(k)$ that is used in the one-dimensional problem (cf. [2, 5]). To improve the positive results, one has to get better estimates on the functions $g_n(k)$, while to improve the negative result, even the signs of these functions have to be considered. A deeper understanding of the problem depends on a generalization of the combinatorial Möbius functions $\mu_{P\times P}$ where P is a locally finite poset. The idea is that the associated zeta function can take any complex value, not merely 0 and 1. We will show elsewhere that Möbius inversion in this context is not appreciably more difficult then what Rota describes in [11]. In this way we hope to attack the problem of more general w_n^k, where perhaps w_n^k is the kth root of an nth degree polynomial. Further studies on this project will be deferred to a later date. We note that question (a) posed in [8] has now been answered, and problems (e) and (f) posed in [8] have also been partially solved in this paper.

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